Web Appendix 1: Details of model analysis: Equilibria and stability analysis

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Our model (Eqs. 3, 4) has three equilibria. The trivial equilibrium ($S^* = 0$, $I^* = 0$) is unstable (a saddle) as long as maximum per capita birth rate (b) of susceptibles (S) exceeds the sum of the two causes of mortality, predation (m_S) and death from other causes (n). The boundary equilibrium ($S_b^* > 0$, $I_b^* = 0$) is defined by

$$S_b^* = K\left(1 - \frac{n + m_S}{b}\right) \qquad (A1)$$

At this equilibrium, density of susceptibles is directly proportional to carrying capacity (K) of the ecosystem. This equilibrium is stable in a system without parasites because of negative density dependence resulting from its logistic form. More specifically, the effect of S on its own growth rate (in a system without infecteds) is a negative function of S, since

$$\frac{\delta(dS/dt)}{\delta S} = -(n + m_S) + b\left(1 - \frac{2S}{K}\right) \qquad (A2)$$

is always negative as long as the boundary equilibrium is feasible (i.e., $b > n + m_s$). Consequently, the one-dimension system has a negative eigenvalue, and, therefore, it is locally stable. This (local asymptotic) stability is important because it means that the invasion/feasibility criterion for the infected class in this model (Eq. 7) is not influenced by cycling behavior, which can make successful invasion of the parasite more difficult (Abrams 1999).

In an ecosystem with both susceptibles and parasites, a stable coexistence equilibrium ($S_i^* > 0$, $I_i^* > 0$) is possible when the parasite can successfully invade a S-only system. We derived such conditions by solving $1/I$ (dI/dt) $= 0$, evaluated at the S-only boundary equilibrium (Eq. A1), S_b^*. This criterion, solved for m_S, is found in Eq. 7. When the parasite successfully invades this S-only system, an interior equilibrium subsequently arises:

$$S_i^* = \frac{n + v}{\beta} + \left(\frac{\alpha}{1 - \alpha}\right)\frac{m_S}{\beta}$$

$$I_i^* = \frac{b(S_b^* - S_i^*)}{b + \beta K} \qquad (A3)$$

It is straightforward to show that this invasion criterion (Eq. 7) is equivalent to the feasibility criterion for the interior equilibrium (i.e., $I_i^* > 0$ as long as Eq. 7 is met). At this equilibrium, susceptible density does not increase with carrying capacity as at the boundary equilibrium. Instead, its density is determined by the ratio of death rates of infecteds

(numerator) to the transmission rate of the parasite (β). Density of the infected class increases with carrying capacity (since $\partial I_i^*/\partial K > 0$) but at a decelerating rate.

It is straightforward to show that, when feasible, this interior equilibrium is also (local asymptotically) stable. When evaluated at this equilibrium (Eq. A3), the Jacobian matrix (**J**) associated with our system (Eq. 3, 4) becomes

$$\mathbf{J} = \begin{bmatrix} -\dfrac{b}{\beta K}\left(\dfrac{\alpha m_S}{1 - \alpha}\right) - \dfrac{n + v}{\beta K} & -\dfrac{(b + \beta K)S_i^*}{K} \\ \beta I_i^* & 0 \end{bmatrix} \qquad (A4)$$

J has negative elements J_{11} and J_{12} and positive element J_{21}. A two-dimensional matrix like **J** has a characteristic polynomial $\lambda^2 + A_1\lambda + A_2$, where λ are the eigenvalues. According to the well-known Routh-Hurwitz criteria (Kot 2001), the model is stable if $A_1 = -J_{11} > 0$, which is always met, and if $A_2 = -J_{12}J_{21} > 0$, which is also always met. Therefore, for our system, a feasible interior equilibrium is always stable.

In addition, the invasion/feasibility criterion (Eq. 7) is always a negative function of predator selectivity upon infecteds (α), but it can be either concave up or down. This criterion (Eq. 7) is always negative because its partial derivative with respect to α

$$\frac{\partial \hat{m}_S}{\partial \alpha} = -\frac{b[b(\beta K - n - v) - d\beta K]}{[\beta K + \alpha(b - \beta K)]^2} \qquad (A5)$$

is always negative when the interior equilibrium is feasible. This negatively sloped relationship is concave down ($\partial^2 m_S/\partial \alpha^2 < 0$) when

$$\alpha < \frac{\beta K}{\beta K - b} \qquad (A6)$$

Given that α ranges from 0 to 1, Eq. A6 means that the isoclines in Fig. 3 are concave down when $\beta K > b$, concave up when $\beta K < b$ (i.e., when the inequality is reversed in Eq. A6), and linear when $\beta K = b$ (i.e., when the denominator of Eq. A6 equals zero). Incidentally, there are other inflections points for m_S, but these points occur when the interior equilibrium is not feasible.

References

ABRAMS, P. A. 1999. Is predator-mediated coexistence possible in unstable systems? Ecology **80:** 608–621.

KOT, M. 2001. Elements of mathematical ecology. Cambridge University Press.